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GENERIC SINGULARITIES OF SCHUBERT VARIETIES

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Abstract. We describe the generic singularity of a Schubert variety of type A on each irreducible component of its singular locus. This singularity is given either by a cone of rank one matrices, or a quadratic cone.

1. Introduction

Let $\mathbb{F}_n$ be the variety of complete flags of an n-dimensional vector space over an algebraically closed field of characteristic zero. For each reference flag $V_\bullet$, and each permutation $w \in \mathcal{S}_n$, one can define a Schubert variety

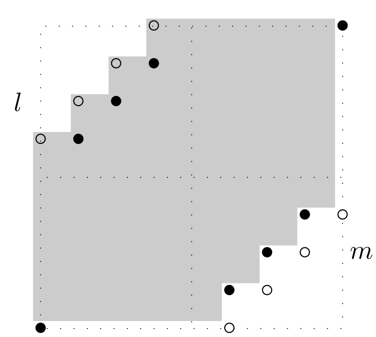
$$X_w = \{W_\bullet \in \mathbb{F}_n,\ \dim(W_p \cap V_q) \geq r_w(p,q),\ 1 \leq p,q \leq n\},$$

where the rank function r_w is defined $r_w(p,q) = \#\{i \leq p,\ w(i) \leq q\}$. This Schubert variety is the Zariski closure of an affine cell Ω_w of dimension $l(w)$, the length of the permutation w.

This note is an appendix to [9], where we solved the longstanding problem of locating the irreducible components of Schubert varieties (see also [1] and [5]). Here we describe geometrically the singularity of X_w at the generic point of each component of its sigular locus. This was done in [3] for *covexillary* permutations (no 3412 configuration allowed), and in a more general but also more restricted context (corresponding in type A to the case of grassmannians rather than complete flag varieties), in [2].

Before stating our result we need to recall how the singular locus of X_w can be located. Recall that following a theorem of Lakshmibai and Sandhya, X_w is singular if and only if there is no sequence of integers $i < j < k < l$ such that $w(l) < w(j) < w(k) < w(i)$ (a 4231 configuration) or $w(k) < w(l) < w(i) < w(j)$ (a 3412 configuration). In general, the irreducible components of $\mathrm{Sing}(X_w)$ are in correspondance with certain minimal 4213 or 3412 configurations encoded in the following figures, where the $\bullet$ are points of the diagram of w (their coordinates are $(i, w(i))$ for some i). There are actually three types.

Type 4231 : $l(w) - l(v) = l + m + 1$, $m(w,v) = l(w) + lm = l(v) + (l+1)(m+1)$, where $l > 0$ and $m > 0$ are the numbers of $\bullet$ in the NorthWest and SouthEast squares, respectively.



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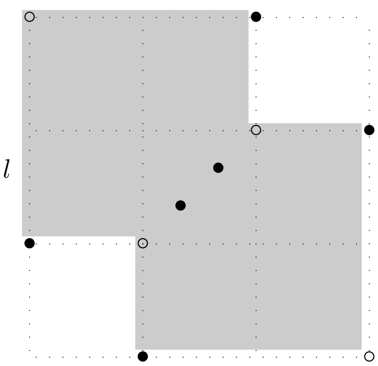
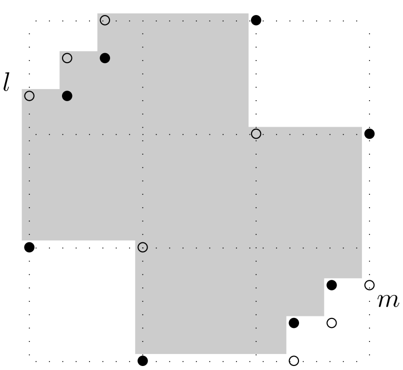
Type 34∗12 : $l(w) - l(v) = 2l + 3$, $m(w, v) = l(w) + 1 = l(v) + 2l + 4$, where $l \geq 0$ is the number of $\bullet$ in the central square.



Type 34∅12 : $l(w) - l(v) = l + m + 3$, $m(w, v) = l(w) + l + m + 1 = l(v) + 2(l + m + 2)$, where $l \geq 0$ and $m \geq 0$ are the numbers of $\bullet$ in the NorthWest and SouthEast squares, respectively.



The configurations formed by the $\bullet$ in the figures above are minimal if there is no other point of the diagram of w in the region D drawn in grey. For each such configuration, we then replace the $\bullet$ by the points represented by $\circ$, to obtain the diagram of a new permutation v. Then X_v is an irreducible component of the singular locus of X_w, and every irreducible component is obtained that way. (Moreover, D is precisely the region where $r_v > r_w$.) The dimension $m(w, v)$ of the Zariski tangent space T_xX_w at a point $x \in \Omega_v$ was computed in [9].

Theorem. *Let X_v be an irreducible component of X_w, coming from one of the three possible types of minimal configurations listed above. Then each point of Ω_v has an affine neighbourhood in $\mathbb{F}_n$ whose intersection with X_w is isomorphic to the product of the affine cell Ω_v, of dimension $l(v)$, with either*

1. *a cone of matrices of size $(l + 1) \times (m + 1)$ and rank at most one;*
2. *a quadratic cone of dimension $2l + 3$;*
3. *a cone of matrices of size $2 \times (l + m + 2)$ and rank at most one.*

The following corollary is an immediate consequence of the theorem and of the computations of [2], 3.3. It was obtained in a purely combinatorial way in [1], 12, but our geometric statement is of course more precise:

Corollary. *With the same notations as above, the Kazhdan-Lusztig polynomial of the pair (v, w) is, respectively,*

1. $P_{v,w}(q) = 1 + q + \cdots + q^{\min(l,m)}$;
2. $P_{v,w}(q) = 1 + q^{l+1}$;
3. $P_{v,w}(q) = 1 + q$.

2. Proof of the theorem

As in [2, 3], we use the existence of a transversal $\mathcal{N}_{v,w}$ to Ω_v in X_w, which was noticed in [6], Lemma A.4. This transversal (whose dimension is of course $l(w)-l(v)$) is the intersection of X_w with $\mathcal{N}_v = v(\Omega_{w_0}) \cup \Omega_{w_0}$, where w_0 is the permutation with maximal length in $\mathcal{S}_n$ (Ω_{w_0} is the "big cell" in $\mathbb{F}_n$, isomorphic to the unipotent group U^- of strict lower triangular matrices in GL_n), and the permutation v is identified with its matrix (assuming that the reference flag is just the canonical flag). The columns of v generate a flag $D_v \in \Omega_v$, and the Bruhat decomposition implies that the map $\phi_v : \mathcal{M}_v = vU^- \cap U^- v \longrightarrow \mathcal{N}_v$ given by $\phi_v(m) = m(D_v)$, is an isomorphism. Note that

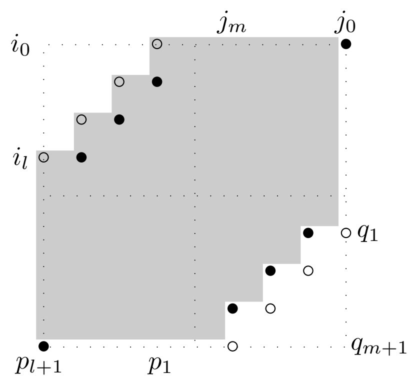
$$\mathcal{M}_v = \{m \in GL_n,\ m_{iv(i)} = 1,\ m_{jk} = 0 \text{ if } k < v(j) \text{ or } j > v^{-1}(k)\}.$$

Lemma. *Let $m \in \phi_v^{-1}(\mathcal{N}_{v,w})$. Let j,k be such that $k > v(j)$ and $j < v^{-1}(k)$. Then $m_{jk} = 0$ as soon as the rectangle $[j, v^{-1}(k)[\times [v(j), k[$ is not contained in the region D where $r_v > r_w$.*

Proof. Choose a basis $e_1, \dots, e_n$ adapted to the reference flag $V_\bullet$. Let $W_\bullet = \phi_v(m)$, with $m \in \mathcal{M}_v$, and $(p,q) \notin D$. Then $W_p + V_q$ is generated by the vectors $e_1, \dots, e_q$ and $m(e_1), \dots, m(e_p)$. Since $m(e_j) = e_{v(j)} + \sum_{k > v(j)} m_{jk} e_k$, the subfamily formed by $e_1, \dots, e_q$ and those $m(e_j)$ such that $j \le p$ and $v(j) > q$ consists in independant vectors. Since there are $q + p - r_v(p,q) = q + p - r_w(p,q)$ of them, we get $dim(W_p \cap V_q) \le r_w(p,q)$. If $W_\bullet$ belongs to X_w, we must have equality, and this implies that the other vectors of the family, that is the $m(e_j)$ for which $j \le p$ and $v(j) \le q$, must be linear combinations of the previous ones. Hence $m_{jk} = 0$ if $j \le p$, $v(j) > q$, and $k > q$ is not among $v(1), \dots, v(p)$, that is $v^{-1}(k) > p$. The lemma follows immediately. □

Now we study our three cases separately.

First case. The first one, that of a minimal 4231 configuration, is completely similar to [3], Théorème 3.6. We fix our notations as in the figure below.



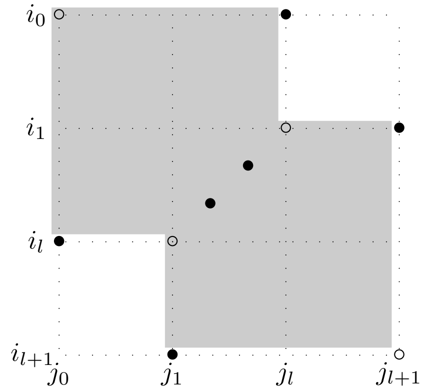
The Lemma implies that a matrix $m \in \phi_v^{-1}(\mathcal{N}_{v,w})$ can have non zero entries, except for those which must be equal to one, only on the lines $j_0, \dots, j_l$ and the columns $k_0, \dots, k_m$. Denote by J and K these sets of indices, this makes an $(l+1) \times (m+1)$ submatrix $m_{J,K}$ of indeterminates.

Now consider the incidence condition corresponding to the point (i_l, p_1), which belongs to D. As in the proof of the Lemma, we see that $W_{i_l} + V_{p_1}$ contains the independant family consisting in the vectors $e_1, \dots, e_{p_1}$ and the $m(e_j)$ for $j \le i_l$ and $v(j) > p_1$. Moreover, since $r_v(i_l, p_1) = r_w(i_l, p_1) + 1$, the dimension of $W_{i_l} + V_{p_1}$ is at most one more than the number of vectors in this family. But $W_{i_l} + V_{p_1}$ also contains $m(e_{i_0}), \dots, m(e_{i_l})$, and in consequence, the rank of M cannot be larger than one. Therefore $\phi_v^{-1}(\mathcal{N}_{v,w}) \subset \mathcal{N}'_{v,w}$, where

$$\mathcal{N}'_{v,w} = \{m \in GL_n, m_{iv(i)} = 1,\ m_{jk} = 0 \text{ if} j \notin J \text{ or } k \notin K,\ \text{rank}(m_{J,K}) \le 1\}.$$

But this is an irreducible variety of the same dimension as $\mathcal{N}_{v,w}$, hence there must be equality.

Second case. We fix our notations as in the figure below.



Here the Lemma implies that a matrix $m \in \phi_v^{-1}(\mathcal{N}_{v,w})$ can have non zero entries, except for those which must be equal to one, only on the line j_0 and the columns $k_1, \dots, k_{l+1}$, or on the column k_{l+2}, and the lines $j_1, \dots, j_{l+1}$.

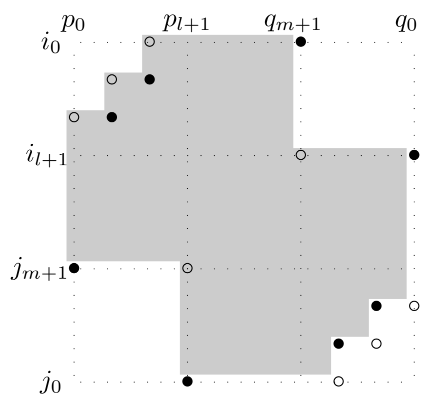
Consider the incidence condition corresponding to the point (j_l, k_0), which does not belong to D. It implies that $W_{j_l} + V_{k_0}$ has a basis consisting of $e_1, \dots, e_{k_0}$ and the $m(e_j)$ such that $j \le j_l$ and $v(j) > k_0$. Then

$$m(e_{j_0}) = e_{k_0} + a_1 e_{k_1} + \cdots + a_{l+1} e_{k_{l+1}}$$

must be a linear combination of these vectors, hence of $m(e_{j_1}) = e_{k_{l+1}} + b_1 e_{k_{l+2}}, \dots, m(e_{j_{l+1}}) = e_{k_1} + b_{l+1} e_{k_{l+2}}$, and e_{k_0}. This is equivalent to the quadratic condition $a_1 b_{l+1} + \cdots + a_{l+1} b_1 = 0$.

All these conditions define a quadratic cone $\mathcal{N}'_{v,w}$ containing $\phi_v^{-1}(\mathcal{N}_{v,w})$. Since it is irreducible of the same dimension as $\mathcal{N}_{v,w}$, again there must be equality.

Third case. This case is slightly more complicated than the previous ones. We fix our notations as in the figure below.



The Lemma implies that a matrix $m \in \phi_v^{-1}(\mathcal{N}_{v,w})$ can have non zero entries, except for those which must be equal to one, only on the lines $i_0 \dots, i_l$ and the columns p_{l+1}, q_{m+1} (giving the coefficients of a $(l+1) \times 2$ matrix A), or on the lines i_{l+1}, j_{m+1} and the columns $q_0, \dots, q_m$ (giving the coefficients of a $2 \times (m+1)$ matrix B).

Exactly as in the first case, the incidence conditions corresponding to the points (i_l, p_l) and (j_{m+1}, q_{m+1}) impose that $\mathrm{rank}(A) \le 1$ and $\mathrm{rank}(B) \le 1$, respectively.

Now consider the incidence condition given by the point (j_{m+1}, p_l), which is not in D. It implies that $W_{j_{m+1}} + V_{p_l}$ has a basis consisting of $e_1, \dots, e_{p_l}$ and the $m(e_j)$ such that $j \le j_{m+1}$ and $v(j) > p_l$. Then $m(e_{i_0}), \dots, m(e_{i_l})$ must be linear combinations of the previous vectors. Thus, more precisely, $m(e_{i_j})$ must be a linear combination of $e_{p_{l-j}}$, $m(e_{i_{l+1}})$ and $m(e_{j_{m+1}})$. If we denote by a, a' the two columns of A, and by b, b' the two lines of B, this means that $a_i b_j + a'_i b'_j = 0$ for all i, j.

All these conditions define an irreducible variety $\mathcal{N}'_{v,w}$ containing $\phi_v^{-1}(\mathcal{N}_{v,w})$. Since it has the same dimension as $\mathcal{N}_{v,w}$, again there must be equality.

Note that if we see the matrices A and B as morphisms $A : k^2 \longrightarrow k^{l+1}$ and $B : k^{m+1} \longrightarrow k^2$, they must have rank at most one and $A \circ B$ must be zero. If we associate to such a pair the $2 \times (l+m+2)$ matrix

$$C = \begin{pmatrix} a_0 & \cdots & a_{l+1} & b'_0 & \cdots & b'_{m+1} \\ a'_0 & \cdots & a'_{l+1} & -b_0 & \cdots & -b_{m+1} \end{pmatrix},$$

we obtain an isomorphism on the cone of rank one matrices, and the proof of the Theorem is complete.

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